Empirical Welfare Analysis with Hedonic Budget Constraints

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Abstract

We analyze demand settings where heterogeneous consumers maximize utility for product attributes subject to a nonlinear budget constraint. We develop nonparametric methods for welfare-analysis of interventions that change the constraint. Two new findings are Roy's identity for smooth, nonlinear budgets, which yields a Partial Differential Equation system, and a Slutsky-like symmetry condition for demand. Under scalar unobserved heterogeneity and single-crossing preferences, the coefficient functions in the PDEs are nonparametrically identified, and under symmetry, lead to path-independent, money-metric welfare. We illustrate our methods with welfare evaluation of a hypothetical change in relationship between property rent and neighborhood school-quality using British microdata.

Keywords: Hedonic model, nonlinear budget, nonparametric identification, welfare, compensating/equivalent variation, partial differential equation, Slutsky symmetry, Roy's Identity, Path Independence.

JEL code: C14, I30, H23

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1 Introduction

Nonlinear budgets arise in a variety of economic applications. A leading example is hedonic modelling of markets for differentiated goods with a large number of available varieties, but where each variety can be viewed as a distinct bundle of a limited number of attributes (Rosen, 1974). Examples include cars, houses, hotels, etc. An important characteristic of such markets is that in equilibrium, the marginal price of an attribute typically varies with quantity, making budget frontiers nonlinear (Diewert, 2003; Ekeland et al 2004). For example, Goodman (1983) records that for new cars, the willingness to pay for additional mileage per gallon (MPG) typically decreases as MPG increases. In empirical models of labour supply, the income tax rate is often progressive, causing potential workers to face piecewise linear budget constraints. Since presence of bunching at kink points is rare in the data, MaCurdy et al. (1990, Section II.D) propose replacing the piecewise linear budget-constraints with a smooth budget frontier to reflect optimization and/or measurement error.[1] The present paper develops econometric methods for welfare analysis of policy interventions in such settings. In contrast to previous research analyzing this problem, we allow for nonparametric, unobserved preference heterogeneity across consumers, and perform exact analysis, as opposed to an approximate one based on linear interpolation of nonlinear budget frontiers (Palmquist, 1988, 2005).

As a motivating example, consider the well-known relationship between housing costs and neighborhood school quality (Sheppard, 1999). To enable children to attend nearby schools, local governments often mandate 'catchment area' rules, which restrict school access solely to neighborhood children. This, however, means that the presence of a good school makes its adjoining residential neighborhood attractive, and raises housing costs. This leads to wealthier families moving in from worse school districts, aggravating existing socioeconomic segregation. One potential way to stop this vicious cycle is to relax catchment area restrictions. This would lead housing costs to become less entangled with school quality, and change choices in equilibrium. For example, Machin and Salvanes (2016) find that relaxing catchment area boundaries in Oslo caused significant weakening of the price-school-quality

[1] We are grateful to Whitney Newey for making us aware of this work.

relation. However, the overall welfare effects of such policy interventions are likely to be heterogeneous, depending on both household preferences over consumption and neighborhood school quality, and on their income. The question is: how can we calculate the distribution of these heterogeneous welfare effects, using microdata on housing and schools. The present paper provides a framework and corresponding econometric methodology to achieve this objective.

In what follows, we present an economic model of choice for a heterogeneous population of consumers, each facing a convex budget set characterized by a nonlinear, smooth frontier. We then derive the analog of Roy's Identity for this setting, which yields a system of linear partial differential equations (PDEs). When unobserved heterogeneity is a scalar, and a single-crossing condition is satisfied by preferences, the coefficient functions of the PDEs can be identified from quantiles of demand. We show that these quantile demand functions must satisfy a Slutsky like symmetry condition which is different from the standard case of linear budget frontiers (Hausman and Newey, 2016). Furthermore, welfare at the quantile can be expressed as a line integral whose value is path-independent under the above symmetry condition. These steps can be repeated separately for each quantile of unobserved heterogeneity to obtain the entire distribution of welfare effects. We emphasize here that the key purpose of this paper is to derive welfare measures, assuming that the budget frontiers are already identified. This means that the relation between price and the attribute of interest (e.g. property rent and neighborhood school quality) is assumed to be known, or consistently estimable from the data. We do not discuss – and indeed, do not contribute to solving – well-known issues of omitted variable problems that jeopardize the identification of this relationship (Black, 1999).

Our work is substantively related to Heckman et al. (2010), who show how to nonparametrically identify consumers' marginal utility for a single continuous attribute in a hedonic setting, where unobserved heterogeneity is a scalar, preferences are quasilinear in consumption and satisfy a single-crossing condition. For the present paper, we borrow the set-up in Heckman et al. (2010), except that utilities are not assumed to be quasilinear in consumption, and our focus is on welfare effects, which we show to be obtainable *without* identifying the underlying marginal utilities, the focus of Heckman et al. (2010). In fact, our paper continues a line of research started by the seminal article of Hausman (1981), subsequently refined in Hausman and Newey (2016), who show that in a demand setting with one continuous inside good, *linear* budget frontiers and general heterogeneity, welfare distributions resulting from a price change are not point-identified. In contrast, our setting allows hedonic budget frontiers to be nonlinear, but (A) restricts heterogeneity to be one-dimensional, and (B) imposes a single-crossing condition, analogous to Heckman et al. (2010). We later show how to include additional attributes into the analysis. Blomquist and Newey (2002) and Blomquist et al. (2021) have investigated identification of demand with general heterogeneity when budget constraints are continuous and piecewise linear, with the slope changing at finitely many kink points.[2]

The rest of the paper is organized as follows. Section 2 describes the set-up and states the key assumptions. Section 3 presents the nonparametric analysis of the problem where functional forms of utilities and how unobserved heterogeneity enter them are not specified. In particular, we show how to obtain the analogs of Roy's identity and Slutsky symmetry in this setting, and how to use the resulting system of PDEs to obtain welfare measures, using data from a large number of markets, each characterized by its own budget frontier. Section 3.3 extends the nonparametric analysis to include additional attributes. Section 4 presents the empirical illustration, and finally, section 5 concludes with directions for future research. All figures and tables are collected at the end of the manuscript, and additional descriptive statistics are reported in an Appendix.

2 Set-up

Denote the key product characteristic by S, a generic value assumed by S to be s, and the hedonic price schedule describing the relation between price and S is given by $P(S) \equiv P(S,\theta)$ where θ is a finite-dimensional parameter. In our empirical illustration, $P(S,\theta)$ is the annual rent for a property whose neighborhood school quality is S. We have data from multiple markets, each with its own θ. For individual consumers, consumption (of the numeraire) is

[2] Once demand distribution is identified for hypothetical linear budget constraints, welfare analysis would resort to methods developed in Hausman and Newey (2016).

given by $C = Y - P(S, \theta)$ where Y is individual income. Individual preferences are described by the utility function $U(S, C, \eta)$ where η represents unobserved preference heterogeneity, and c is consumption. A household maximizes its utility by choosing S optimally, subject to the budget constraint $Y = P(S, \theta) + C$. For the purpose of this paper, viz. identification of welfare effects, we assume that the function $P(S, \theta)$ in each market is known to the analyst,[3] and the marginal (or conditional on observables) distribution of η is identical across markets.

We impose the following assumptions on the utility functions. Let $U_{sc}(s, c, \eta)$, $U_{cc}(s, c, \eta)$, etc. denote the second order derivatives of U.

Assumption 1 *(i) $U(\cdot, \cdot, \eta)$ has continuous second-order derivatives in its first two arguments; (ii) η is a scalar, distributed independently of Y, and is identically distributed in each market, (iii) $U(\cdot, \cdot, \eta)$ is strictly increasing in each argument for any fixed η, (iv) the cross-partial derivatives satisfy $U_{s\eta}(s, c, \eta) > 0$ and $U_{c\eta}(s, c, \eta) \leq 0$ for all values of s, c, η on the support of $(S, Y - P(S, \theta), \eta)$; (v) $P(s, \theta)$ is smooth in both s and θ and is increasing in s for fixed θ; (vi) for all s, y, θ, η, we have that*

$$\left\{ \begin{array}{l} U_{ss}(s, y - P(s;\theta), \eta) - 2\frac{\partial}{\partial s} P(s;\theta) \times U_{cs}(s, y - P(s;\theta), \eta) \\ + \left(\frac{\partial P(s;\theta)}{\partial s}\right)^2 \times U_{cc}(s, y - P(s;\theta), \eta) \end{array} \right\} < 0. \tag{1}$$

The smoothness assumption (i) enables us to obtain the key analytical steps for calculating welfare effects; (ii) is the key substantive restriction on unobserved heterogeneity,[4] and implies rank invariance, i.e. the ordering of any two different consumers' demand remains identical across budget frontiers; (iii) is non-satiation in S and consumption, which is intuitive and is a key sufficient condition for our welfare measure, viz. the compensating variation, to be well-defined. Assumption (iv) says that the marginal utility w.r.t. s is strictly increasing, and marginal utility w.r.t. c is decreasing in η. Intuitively, this means that higher

[3] Indeed, θ will typically be estimated from the data, but at a parametric rate, and these estimated θs will be used subsequently as regressors, leading to standard measurement error issues. However, variance of the measurement error in θ is of order $O(n^{-1})$, where n is the number of observations in each market used to estimate θ in that market. Hence replacing θ by its estimate will lead to a very small attenuation bias when n is large. For related discussions, see Heckman et al, (2010), Section 5.

[4] It allows for η to be a single index of multi-dimensional underlying heterogeneity

η types have higher marginal utility w.r.t. s and lower marginal utility w.r.t. c. This implies the so-called 'single crossing' condition, i.e. that the marginal rate of substitution between S and C is increasing in η:

$$\frac{d}{d\eta}\left(\frac{\frac{\partial U(s,c,\eta)}{\partial s}}{\frac{\partial U(s,c,\eta)}{\partial c}}\right) > 0.$$

It will be shown below that assumption (iv) implies that for fixed budget line, demand for S is strictly monotone in η. This creates a 1-to-1 map between quantiles of *observed* demand and quantiles of *unobserved* preference, which is helpful for identifying welfare. Heckman et al. (2010) assume utilities are quasilinear in consumption, so that $U_{\eta c}(s, c, \eta) \equiv 0$; assumption (iv) is therefore a generalization required to cover the more general non-quasilinear case. Assumption (v) says S is a 'desirable' attribute, i.e. consuming more S costs more. Finally, assumption (vi) says that the hedonic budget frontier should be 'less convex' to the origin than the indifference curves,[5] which guarantees that utility is maximized uniquely at an interior point on the budget frontier. In particular, (1) holds if the budget frontier is strictly concave and indifference curves are strictly convex to the origin.

Remark 1 *Note that, other than smoothness, we make no functional form assumption on utilities, on how they depend on η, or the marginal distribution of η. In particular, we do not require utilities to be increasing in η.*

The policy intervention we wish to evaluate is one that changes the hedonic price frontier. In our empirical illustration, an important case of interest is where school choice becomes less or more restrictive, which would weaken (respectively, strengthen) the relationship between rent and school-quality (Machin and Salvanes, 2016). The pre- and post-intervention situations are depicted via Figure 1 where, for ease of exposition, η is held fixed.

In Figure 1, consumption is measured on the vertical axis, and school quality along the horizontal axis. The original budget frontier $C = Y - P(S, \theta)$ is depicted by the steeper blue curve OD. Utility is maximized at C where the indifference curve, convex to the origin is tangent to OD. Now, due to a policy intervention (e.g. relaxed school choice in our example),

[5] The slope of the indifference curves in the $S - C$ axes are given by $-\frac{U_S}{U_C}$, whereas the budget curve has slope $-p'$. Then (1) is equivalent to the difference between $-\frac{U_S}{U_C} - (-p') = p' - \frac{U_S}{U_C}$ being strictly negative, i.e. the indifference curves are more convex than the budget frontier.

the hedonic price schedule changes, and the budget frontier shifts to the flatter orange curve AE, whence optimal choice is B, representing a fall in utility relative to C.

We wish to compute the welfare effect of this intervention via the compensating variation, which calculates how much would a household need to be compensated, so that its maximized utility with the additional income in the post-intervention situation equals its maximized utility in the pre-intervention period with the original income. To see this graphically, consider the curve depicted by the dashed curve GF, which is the AE translated vertically up and is tangent to the original indifference curve at F. Then the compensating variation, GA>0, is the income supplement needed for the individual facing the blue budget curve so that she can reach utility equal to what she was enjoying initially.

Given the position of the indifference curves, the CV is positive, indicating that the consumer is losing as a result of the change, and hence needs to be compensated by a positive income transfer to restore her utility to its pre-intervention state. However, if the original indifference curve were tangent to OD at a point below its intersection with AE, then the shift of the budget line to AE would lead to a *gain* in utility. Such a consumer would benefit from the change, and the CV will be negative. Intuitively speaking, the former type of households value school quality less relative to consumption, and so were initially consuming relative lower quality schooling. After the intervention, housing costs rise for lower quality school areas, and therefore these households can afford less consumption than before. The latter type of household values school-quality relatively more, and choose higher school quality. The intervention makes housing costs lower for areas with good schools, and hence expands the budget set of these types of consumers. This reasoning illustrates that welfare effects of a shift in the budget frontier can be heterogeneous in both magnitude and sign; hence it is of interest to find the distribution of welfare as the heterogeneity varies across consumers. We now turn to developing the methods for these nonparametric calculations.

3 Demand and Welfare Analysis

In this section, we first derive the analogs of Roy's Identity and Slutsky symmetry, and then move on to show how to identify welfare effects of a change in the budget frontier. We start with the case where there is a single attribute S, and then extend the analysis to include additional attributes.

3.1 Roy's Identity and Slutsky-Symmetry

For ease of exposition, consider the case where S is the only attribute of interest, the known hedonic price function is given by $P(S, \theta)$, where θ is an unknown vector of parameters. We will introduce additional attributes later in Sec 3.3. The utility of an η-type consumer is given by $U(s, c, \eta)$ where y represents disposable income, s is the amount of S chosen, c is consumption of the non-S numeraire, and η is unobserved heterogeneity. Utility maximization and nonsatiation (assumption (iii)) imply that at the optimal choice $S^*(y, \theta, \eta)$, we must have that

$$U_s(s, y - P(s; \theta), \eta) - U_c(s, y - P(s; \theta), \eta) \frac{\partial}{\partial s} P(s; \theta)\bigg|_{s=S^*(y,\theta,\eta)} = 0 \qquad (2)$$

Finally, the indirect utility function is given by

$$V(y, \theta, \eta) = U(S^*(y, \theta, \eta), y - P(S^*(y, \theta, \eta); \theta), \eta). \qquad (3)$$

Then for fixed θ, η, and given assumption 1(i), we have that $V(\cdot, \theta, \eta)$ is differentiable, and the envelope theorem condition holds, i.e.

$$\begin{aligned}
& \frac{\partial V(y,\theta,\eta)}{\partial y} \\
= \; & U_c\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \\
& + U_s\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right)\frac{\partial S^*(y,\theta,\eta)}{\partial y} \\
& - U_c\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \times \frac{\partial}{\partial s} P\left(S^*(y,\theta,\eta);\theta\right)\frac{\partial S^*(y,\theta,\eta)}{\partial y} \\
= \; & \underbrace{\left[\begin{array}{l} U_s\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \\ -U_c\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \times \frac{\partial}{\partial s} P\left(S^*(y,\theta,\eta);\theta\right) \end{array}\right]}_{=0, \text{ by } (2)} \frac{\partial S^*(y,\theta,\eta)}{\partial y} \\
& + U_c\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \\
= \; & U_c\left(S^*(y,\theta,\eta), y - P\left(S^*(y,\theta,\eta);\theta\right)\right) \qquad (4)
\end{aligned}$$

Therefore, $V(\cdot,\theta,\eta)$ is strictly increasing, by assumption 1(iii).

Further, letting $P_j\left(s^*(y,\theta,\eta);\theta\right) = \left.\frac{\partial P(s;\theta)}{\partial \theta_j}\right|_{s=s^*(y,\theta,\eta)}$, we have by the envelope theorem that

$$\frac{\partial V(y,\theta,\eta)}{\partial \theta_j} = -U_c\left(s^*(y,\theta,\eta), y - P\left(s^*(y,\theta,\eta);\theta\right)\right) \times P_j\left(s^*(y,\theta,\eta);\theta\right). \qquad (5)$$

From (4) and (5), it follows that for each $j = 1, 2, ..., \dim(\theta)$, it must hold that

$$-\frac{\frac{\partial V(y,\theta,\eta)}{\partial \theta_j}}{\frac{\partial V(y,\theta,\eta)}{\partial y}} = P_j\left(s^*(y,\theta,\eta);\theta\right) \qquad (6)$$

which can be interpreted as Roy's identity for a nonlinear budget frontier.

Now, suppose we want to measure welfare-effects resulting from a change in θ from a to b. A common money-metric measure is the compensating variation $C \equiv C(y,\eta)$, which solves

$$V(y+C,b,\eta) = V(y,a,\eta). \qquad (7)$$

There is a unique solution in C, since $\frac{\partial V(y,\theta,\eta)}{\partial y} > 0$ with probability 1, by (4).

To find the distribution of C, suppose, initially, that we know the value of η, then we can learn $P_j\left(s^*(y,\theta,\eta);\theta\right)$ from the hedonic price schedule in the data. Now, equation (6) can be rewritten as a system of linear, first-order partial differential equations of order 1

$$\frac{\partial V(y,\theta,\eta)}{\partial \theta_j} + \frac{\partial V(y,\theta,\eta)}{\partial y} \times P_j(s^*(y,\theta,\eta);\theta) = 0. \tag{8}$$

Therefore, the goal is to solve for (7), where $V(\cdot)$ satisfies (8). The key difficulty in calculating welfare effects nonparametrically is that η is unobserved. To address this problem, we use the single-crossing condition and scalar heterogeneity to implement a quantile-based analysis, as follows.

Quantile-based Analysis: If η is a scalar and $S^*(y,\theta,\eta)$ is strictly monotone and invertible in η, then we can interpret the observed τth quantiles of $S^*(y,\theta,\eta)$, conditional on y and θ (cf. assumption (ii) above) as the demand of the individual who is located at the τth quantile of the distribution of η. This is identified by the τth quantile of demand for those at income y on the budget frontier $P(s;\theta)$, i.e.

$$S^*\left(y,\theta,F_\eta^{-1}(\tau)\right) = F_{S^*(y,\theta,\eta)}^{-1}(\tau) \equiv q^\tau(y,\theta),$$

where $\tau \in [0,1]$, and $q^\tau(y,\theta) \equiv F_{S^*(y,\theta,\eta)}^{-1}(\tau)$ equals the τth quantile of demand for those with income y and facing a budget frontier characterized by θ. Further, since the indirect utility function

$$V(y,\theta,\eta) = \max_{s,c} U(s,c,\eta) \text{ s.t. } c = y - P(s,\theta),$$

by the envelope theorem, we have that

$$\frac{\partial}{\partial y} V(y,\theta,\eta) = \frac{\partial U}{\partial c}(S^*(y,\theta,\eta), y - P(S^*(y,\theta,\eta),\theta)) > 0 \tag{9}$$

when utility is strictly increasing in consumption, i.e. assumption (iii).

Now, differentiating the LHS of (2), we get that

$$\left.\begin{array}{l}\left\{U_{ss}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)-U_{sc}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{\partial P}{\partial s}\right\}\frac{dS^*}{d\eta}\\+U_{s\eta}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)-U_{c}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{\partial^2 P\left(S^*;\theta\right)}{\partial s}\frac{dS^*}{d\eta}\\-\frac{\partial}{\partial s}P\left(S^*;\theta\right)\times U_{cs}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{dS^*}{d\eta}\\+\left\{\frac{\partial}{\partial s}P\left(S^*;\theta\right)\right\}^2\times U_{cc}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{dS^*}{d\eta}\\-\frac{\partial}{\partial s}P\left(S^*;\theta\right)\times U_{c\eta}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\end{array}\right| = 0,$$

implying

$$\frac{dS^*}{d\eta}=-\frac{\begin{array}{l}U_{s\eta}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\\-\frac{\partial}{\partial s}P\left(S^*;\theta\right)\times U_{c\eta}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\end{array}}{\begin{array}{l}U_{ss}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\\-U_{c}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{\partial^2 P\left(S^*;\theta\right)}{\partial s^2}\\\left\{\frac{\partial}{\partial s}P\left(S^*;\theta\right)\right\}^2\times U_{cc}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{\partial}{\partial s}P\left(S^*;\theta\right)\\-2U_{sc}\left(S^*,y-P\left(S^*;\theta\right),\eta\right)\frac{\partial}{\partial s}P\left(S^*;\theta\right)\end{array}}$$

The denominator of this expression is negative by (1). The numerator is positive by assumptions (iv) and (v). Hence $\frac{dS^*}{d\eta}>0$ with probability 1. Note that Heckman et al. (2010) derived an analogous result for the case where utility is quasilinear in consumption, so that $U_{\eta c}\left(s,c,\eta\right)=0$, which is a special case of our set-up. In any case, the monotonicity of S^* w.r.t. η will be used below for identifying the distribution of the compensating variation.

In order to implement our method of welfare analysis, it is also useful to introduce a Slutsky-symmetry type result. This result is of independent interest, as it characterizes demand when budget frontiers are nonlinear.

Toward that end, define

$$Q_\tau\left(y,\theta\right)\equiv V\left(y,\theta,F_\eta^{-1}\left(\tau\right)\right), \tag{10}$$

i.e. the indirect utility obtained by an individual with income y and located at the τth quantile of unobserved heterogeneity, i.e. whose value of η equals $F_\eta^{-1}\left(\tau\right)$, when the price function is characterized by the parameter θ.

Lemma 1 (Slutsky-symmetry for nonlinear budget-sets) *Suppose the price-attribute relationship is given by $P\left(s,\theta\right)$ where θ is of dimension $d\geq 2$. Let $q^\tau\left(y,\theta\right)$ denote the demand* *at the τth quantile of η when income is fixed at y. Then for each $j, k \in \{1, 2, ...d\}$, it holds that*

$$\begin{aligned} &\tfrac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial\theta_j \partial q}\left\{\tfrac{\partial q^\tau(y,\theta)}{\partial y}\tfrac{\partial P(q^\tau(y,\theta),\theta)}{\partial\theta_k}+\tfrac{\partial q^\tau(y,\theta)}{\partial\theta_k}\right\} \\ &= \tfrac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial\theta_k \partial q}\left\{\tfrac{\partial q^\tau(y,\theta)}{\partial y}\tfrac{\partial P(q^\tau(y,\theta),\theta)}{\partial\theta_j}+\tfrac{\partial q^\tau(y,\theta)}{\partial\theta_j}\right\} \end{aligned} \tag{11}$$

Proof. Let $e(\boldsymbol{\theta}, u)$ denote the expenditure function, i.e. the solution to

$$Q_\tau(\boldsymbol{\theta}, e) = u.$$

This function is well defined since $Q_\tau(\boldsymbol{\theta}, e)$ is continuous and strictly increasing in e. Let $j = 1$ and $k = 2$ WLOG. Then, by definition

$$\begin{aligned} &\tfrac{\partial}{\partial\theta_1}\{Q_\tau(\boldsymbol{\theta}, e(\boldsymbol{\theta}, u))\} = 0 \\ &\implies \tfrac{\partial}{\partial\theta_1}Q_\tau(\boldsymbol{\theta}, e(\boldsymbol{\theta}, u)) + \tfrac{\partial}{\partial e}Q_\tau(\boldsymbol{\theta}, e(\boldsymbol{\theta}, u))\tfrac{\partial e(\boldsymbol{\theta},u)}{\partial\theta_1} = 0 \\ &\implies \tfrac{\partial e(\boldsymbol{\theta},u)}{\partial\theta_1} = -\frac{\frac{\partial}{\partial\theta_1}Q_\tau(\boldsymbol{\theta},e(\boldsymbol{\theta},u))}{\frac{\partial}{\partial e}Q_\tau(\boldsymbol{\theta},e(\boldsymbol{\theta},u))} \overset{\text{by (20)}}{=} \left.\tfrac{\partial P(s,\theta)}{\partial\theta_1}\right|_{s=q^\tau(e(\boldsymbol{\theta},u),\theta)} \end{aligned}$$

Similarly,

$$\frac{\partial e(\boldsymbol{\theta}, u)}{\partial\theta_2} = \left.\frac{\partial P(s, \theta)}{\partial\theta_2}\right|_{s=q^\tau(e(\boldsymbol{\theta},u),\theta)}$$

Thus we have that

$$\frac{\partial e(\boldsymbol{\theta}, u)}{\partial\theta_1} = \left.\frac{\partial P(s, \theta)}{\partial\theta_1}\right|_{s=q^\tau(e(\boldsymbol{\theta},u),\theta)} \tag{12}$$

$$\frac{\partial e(\boldsymbol{\theta}, u)}{\partial\theta_2} = \left.\frac{\partial P(s, \theta)}{\partial\theta_2}\right|_{s=q^\tau(e(\boldsymbol{\theta},u),\theta)} \tag{13}$$

Now, differentiating (12) w.r.t. θ_2, we get

$$\begin{aligned} \frac{\partial^2 e(\boldsymbol{\theta}, u)}{\partial\theta_2\partial\theta_1} &= \frac{\partial}{\partial\theta_2}\left\{\frac{\partial P(q^\tau(e(\boldsymbol{\theta}, u), \theta), \theta)}{\partial\theta_1}\right\} \\ &= \frac{\partial^2 P(q^\tau(e(\boldsymbol{\theta}, u), \theta), \theta)}{\partial\theta_2\partial q}\left\{\begin{array}{l} \frac{\partial q^\tau(e(\boldsymbol{\theta},u),\theta)}{\partial y}\frac{\partial P(q^\tau(e(\boldsymbol{\theta},u),\theta),\theta)}{\partial\theta_1} \\ +\frac{\partial q^\tau(e(\boldsymbol{\theta},u),\theta)}{\partial\theta_1} \end{array}\right\} \\ &\quad +\left\{\frac{\partial^2 P(q^\tau(e(\boldsymbol{\theta}, u), \theta), \theta)}{\partial\theta_1\partial\theta_2}\right\} \end{aligned} \tag{14}$$

Similarly, differentiating (13) w.r.t. θ_1, we get

$$\frac{\partial^2 e(\boldsymbol{\theta}, u)}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 P(q^\tau(e(\boldsymbol{\theta}, u), \theta), \theta)}{\partial \theta_1 \partial q} \left\{ \begin{array}{l} \frac{\partial q^\tau(e(\boldsymbol{\theta},u),\theta)}{\partial y} \frac{\partial P(q^\tau(e(\boldsymbol{\theta},u),\theta),\theta)}{\partial \theta_2} \\ + \frac{\partial q^\tau(e(\boldsymbol{\theta},u),\theta)}{\partial \theta_2} \end{array} \right\} + \left\{ \frac{\partial^2 P(q^\tau(e(\boldsymbol{\theta}, u), \theta), \theta)}{\partial \theta_1 \partial \theta_2} \right\} \tag{15}$$

Equating (14) and (15), and evaluating at $y = e(\boldsymbol{\theta}, u)$, we get the symmetry condition

$$\begin{aligned} & \frac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial \theta_1 \partial q} \left\{ \frac{\partial q^\tau(y,\theta)}{\partial y} \frac{\partial P(q^\tau(y,\theta),\theta)}{\partial \theta_2} + \frac{\partial q^\tau(y,\theta)}{\partial \theta_2} \right\} \\ = & \frac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial \theta_2 \partial q} \left\{ \frac{\partial q^\tau(y,\theta)}{\partial y} \frac{\partial P(q^\tau(y,\theta),\theta)}{\partial \theta_1} + \frac{\partial q^\tau(y,\theta)}{\partial \theta_1} \right\} \end{aligned} \tag{16}$$

■

Equation (16) is the analog of Slutsky symmetry when budget constraints are nonlinear. Indeed, in the standard case with linear budgets, where $P(q^\tau(y, \theta), \theta) = \theta q^\tau(y, \theta)$, (16) would reduce to

$$\frac{\partial q^\tau(y, \theta)}{\partial y} q^\tau(y, \theta) + \frac{\partial q^\tau(y, \theta)}{\partial \theta_2} = \frac{\partial q^\tau(y, \theta)}{\partial y} q^\tau(y, \theta) + \frac{\partial q^\tau(y, \theta)}{\partial \theta_1}$$

which is the textbook case of Slutsky symmetry with linear budget frontiers.

3.2 Calculation of Money-metric Welfare

We now outline the steps for obtaining welfare effects. Toward that end, first note from (9) and (10) that

$$\frac{\partial}{\partial y} Q_\tau(y, \theta) > 0,$$

and from (2) that for all $\tau \in [0, 1]$, and $j = 1, ..., J$,

$$\frac{\partial Q_\tau(y, \theta)}{\partial \theta_j} + \left. \frac{\partial P(s, \theta)}{\partial \theta_j} \right|_{s = q^\tau(y, \theta)} \times \frac{\partial Q_\tau(y, \theta)}{\partial y} = 0. \tag{17}$$

Suppose the parameter θ characterizing the budget frontier changes from a to b, and we wish to calculate the compensating variation corresponding to this change for this individual with $\eta = F_{\eta}^{-1}(\tau)$. The compensating variation is defined as the income supplement required to maintain the utility of the consumer, i.e. solve for $C = C(y, \tau)$ which satisfies

$$Q_\tau(y + C, a) = Q_\tau(y, b), \text{ i.e. } C(y, \tau) = Q_\tau^{-1}(Q_\tau(y, b), a) - y. \tag{18}$$

Now, note that the function $P(s, \theta)$ and the quantile demand $q^\tau(y, \theta)$ are identified from the observed data. Suppose, for concreteness, that the hedonic price function is given by

$$P(S, \theta) = \theta_1 + \theta_2 \ln(S). \tag{19}$$

The values of θ_1, θ_2 vary across markets. Our goal is to find C which solves

$$Q_\tau(y + C, b) = Q_\tau(y, a),$$

where $Q_\tau(\cdot, \theta)$ is strictly increasing, and satisfies the system

$$\begin{array}{l} \frac{\partial Q_\tau(y,\theta_1,\theta_2)}{\partial \theta_1} + \frac{\partial Q_\tau(y,\theta_1,\theta_2)}{\partial y} = 0, \\ \frac{\partial Q_\tau(y,\theta_1,\theta_2)}{\partial \theta_2} + \ln\{q^\tau(y, \theta)\} \times \frac{\partial Q_\tau(y,\theta_1,\theta_2)}{\partial y} = 0. \end{array} \tag{20}$$

The term $\ln q^\tau(y, \theta)$ can be identified from the data by running a quantile regression of (natural log of) the demanded attribute on individual income and the market level θ, when there are multiple markets, each with its own θ. It is natural to start with the simple linear specification for the quantile regression function (which may be generalized to a higher order polynomial, spline, etc.)

$$\ln q^\tau(y, \theta) = r_0 + r_1 y + r_2 \theta_1 + r_3 \theta_2, \tag{21}$$

where the r coefficients are obtained via a τ-quantile regression of $\ln S$ on individual y and market-level θ_1 and θ_2. Before proceeding further, it is important to verify that (21) is indeed a valid specification for demand.

Proposition 1 *In order for (21) to be a valid specification for demand, it is necessary that* $r_1 + r_2 = 0$.

Proof of proposition 2. From (12), (19), (21), we have that

$$\frac{\partial e(\boldsymbol{\theta}, u)}{\partial \theta_1} \overset{\text{by (19)}}{=} 1 \Longrightarrow \frac{\partial^2 e(\theta_1, \theta_2, u)}{\partial \theta_2 \partial \theta_1} = 0. \tag{22}$$

On the other hand, by (13)

$$\begin{aligned} \tfrac{\partial e(\boldsymbol{\theta}, u)}{\partial \theta_2} &\overset{\text{by (19)}}{=} \ln q^{\tau}(\theta_1, \theta_2, e(\theta_1, \theta_2, u)) \\ &\overset{\text{by (21)}}{=} r_0 + r_1 \times e(\theta_1, \theta_2, u) + r_2\theta_1 + r_3\theta_2 \\ \Longrightarrow \quad \frac{\partial^2 e(\theta_1, \theta_2, u)}{\partial \theta_1 \partial \theta_2} &= r_2 + r_1 \frac{\partial e(\theta_1, \theta_2, u)}{\partial \theta_1} \overset{\text{by (19)}}{=} r_2 + r_1. \end{aligned} \tag{23}$$

Now (22) and (23) imply that $r_1 + r_2 = 0$. ■

Now suppose the value of the parameter θ characterizing the price frontier changes from a to b. We wish to find the compensating variation, which is the hypothetical income transfer that an individual needs when $\theta = b$ to be able to reach the utility level she had attained when θ equalled a. Given heterogeneous preferences, the compensating variation for the same change in θ is heterogeneous, and we wish to obtain its distribution. To do this, we fix a value of $\tau \in [0, 1]$, and develop a method to compute the CV for individuals whose $\eta = F_{\eta}^{-1}(\tau)$. We then vary τ to generate the CV for different quantiles of η.

Toward that end, consider a price path $\theta(t)$, with $t \in [0, 1]$, such that $\theta_1(0) = a_1$, $\theta_2(0) = a_2$, $\theta_1(2) = b_1$, $\theta_2(1) = b_2$. By definition, the compensating variation at a generic value of $t \in [0, 1]$ is given by $C(t, y) = e(\theta(t), \bar{u}) - y$, will satisfy

$$Q_{\tau}(\theta(t), y + C(t, y)) = \bar{u} \text{ for all } t, \tag{24}$$

where the initial utility level $\bar{u} = Q_{\tau}(a_1, a_2, y)$. Differentiating (24) w.r.t. t we have that for all t,

$$\frac{d}{dt} Q_{\tau}(\theta(t), y + C(t, y)) = 0, \text{ i.e.}$$

for all t,

$$\sum_j \frac{\partial Q_{\tau}(\theta(t), y + C(t, y))}{\partial \theta_j} \frac{d\theta_j(t)}{dt} + \frac{\partial Q_{\tau}(\theta(t), y + C(t, y))}{\partial y} \frac{dC(t, y)}{dt} = 0.$$

This implies

$$\begin{aligned}\frac{dC(t,y)}{dt} &= -\sum_j \frac{\frac{\partial Q_\tau(\theta(t),y+C(t,y))}{\partial\theta_j}}{\frac{\partial Q_\tau(\theta(t),y+C(t,y))}{\partial y}} \frac{d\theta_j(t)}{dt} \\ &\overset{\text{by (17)}}{=} \sum_j \frac{d\theta_j(t)}{dt} \times \left.\frac{\partial P(\theta,s)}{\partial\theta_j}\right|_{\theta=\theta(t),s=q_\tau(\theta(t),e(\theta(t),\bar{u}))} \end{aligned} \tag{25}$$

The second term on the RHS of (25) is identifiable from the data across many markets, since $e(\theta(t),\bar{u}) = y + C(t,y)$. So finding the compensating variation for a change in θ from a to b for an individual at income y and whose η equals its τth quantile is equivalent to finding $C(1,y)$, where $C(t,y)$ solves (25) with the initial condition $C(0,y) = 0$. Observe that

$$\begin{aligned} C(1.y) - C(0,y) &= \int_0^1 \frac{dC(t,y)}{dt} dt \\ &= \int_0^1 \left\{ \sum_j \frac{d\theta_j(t)}{dt} \times \frac{\partial P(\theta(t), q_\tau(\theta(t), e(\theta(t),\bar{u})))}{\partial\theta_j} \right\} dt \\ &= \int_{\mathcal{G}} \sum_j \frac{\partial P(\theta, q_\tau(\theta, e(\theta,\bar{u})))}{\partial\theta_j} d\theta_j. \end{aligned} \tag{26}$$

The final expression on the RHS is a line integral of the vector field $\left(\frac{\partial P(\theta,q_\tau(\theta,e(\theta,\bar{u})))}{\partial\theta_1}, \frac{\partial P(\theta,q_\tau(\theta,e(\theta,\bar{u})))}{\partial\theta_2}\right)$ along the path $\mathcal{G} = \{\theta_1(t), \theta_2(t)\}$, $t \in [0,1]$ connecting the points (a_1,a_2) and (b_1,b_2). The symmetry condition (16) and the gradient theorem for line integrals (cf. Spiegel 2010, Sec 10.6; Courant and John 1989, Sec 1.10) then imply that the value of (26) is path-independent, i.e. its value does not depend on the path $\mathcal{G}$. Thus the compensating variation $C(1.y)$ is well-defined.

In particular, given the specification (19) and (21), equation (25) reduces to the ordinary differential equation

$$
\begin{aligned}
& \frac{dC(t,y)}{dt} \\
& \overset{\text{(by 21)}}{=} \frac{d\theta_1(t)}{dt} + \frac{d\theta_2(t)}{dt} \times \left(r_0 + r_1 \left(y + C(t,y)\right) + r_2\theta_1(t) + r_3\theta_2(t)\right) \\
& \overset{\text{by } r_2+r_1=0}{=} \frac{d\theta_1(t)}{dt} + \frac{d\theta_2(t)}{dt} \times \left(r_0 + r_1 \left(y + C(t,y) - \theta_1(t)\right) + r_3\theta_2(t)\right) \\
\Leftrightarrow \quad & \frac{dC(t,y)}{dt} - r_1 \frac{d\theta_2(t)}{dt} \times C(t,y) \\
= \quad & \frac{d\theta_1(t)}{dt} + \frac{d\theta_2(t)}{dt} \times \left(r_0 + r_1 \left(y - \theta_1(t)\right)\right) + r_3 \frac{d\theta_2(t)}{dt} \times \theta_2(t)
\end{aligned}
$$

This implies

$$
\begin{aligned}
& \frac{dC(t,y)}{dt} - r_1 \frac{d\theta_2(t)}{dt} \times C(t,y) \\
= \quad & \frac{d\theta_1(t)}{dt} + \frac{d\theta_2(t)}{dt} \times \left(r_0 + r_1 \left(y - \theta_1(t)\right)\right) + r_3 \frac{d\theta_2(t)}{dt} \times \theta_2(t)
\end{aligned}
$$

This linear ODE can be solved using the method of integrating factors as

$$
\begin{aligned}
& \frac{dC(t,y)}{dt} e^{-r_1\theta_2(t)} - r_1 \frac{d\theta_2(t)}{dt} e^{-r_1\theta_2(t)} \times C(t,y) \\
= \quad & \frac{d\theta_1(t)}{dt} e^{-r_1\theta_2(t)} + e^{-r_1\theta_2(t)} \frac{d\theta_2(t)}{dt} \times \left(r_0 + r_1 \left(y - \theta_1(t)\right)\right) \\
& + r_3 \frac{d\theta_2(t)}{dt} \times \theta_2(t) e^{-r_1\theta_2(t)} \\
= \quad & \frac{d\theta_1(t)}{dt} e^{-r_1\theta_2(t)} - e^{-r_1\theta_2(t)} \frac{d\theta_2(t)}{dt} \times \theta_1(t) \\
& + \left(r_0 + r_1 y\right) e^{-r_1\theta_2(t)} \frac{d\theta_2(t)}{dt} + r_3 \frac{d\theta_2(t)}{dt} \times \theta_2(t) e^{-r_1\theta_2(t)}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{d}{dt}\left\{C(t,y) e^{-r_1\theta_2(t)}\right\} \quad = \quad & \frac{d}{dt}\left\{\theta_1(t) e^{-r_1\theta_2(t)}\right\} \\
& + e^{-r_1\theta_2(t)} \frac{d\theta_2(t)}{dt} \left(r_0 + r_1 y\right) \\
& + r_3 \frac{d\theta_2(t)}{dt} \times \theta_2(t) e^{-r_1\theta_2(t)}
\end{aligned}
$$

Integrating both sides, we get that

$$\begin{aligned} C(t,y)\, e^{-r_1\theta_2(t)} &= Cons + \theta_1(t)\, e^{-r_1\theta_2(t)} - (r_0 + r_1 y)\frac{e^{-r_1\theta_2(t)}}{r_1} \\ &\quad -r_3\theta_2(t)\frac{e^{-r_1\theta_2(t)}}{r_1} - \frac{e^{-r_1\theta_2(t)}}{r_1^2} r_3, \end{aligned} \tag{27}$$

where "$Cons$" denotes a constant. This implies

$$C(t,y) = Cons \times e^{r_1\theta_2(t)} + \theta_1(t) - (r_0 + r_1 y)\frac{1}{r_1} - r_3\theta_2(t)\frac{1}{r_1} - \frac{r_3}{r_1^2}$$

Applying the boundary condition $C(0,y) = 0$, and $\theta_1(0) = a_1$, $\theta_2(0) = a_2$, we get

$$Cons = e^{-r_1 a_2}\left\{-a_1 + (r_0 + r_1 y)\frac{1}{r_1} + a_2\frac{r_3}{r_1} + \frac{r_3}{r_1^2}\right\}.$$

Replacing in (27), and evaluating at $t = 1$, using $\theta_1(1) = b_1$, $\theta_2(1) = b_2$, we get

$$\begin{aligned} C(1,y) &= e^{r_1(b_2-a_2)}\left\{-a_1 + \frac{r_0}{r_1} + y + \frac{a_2 r_3}{r_1} + \frac{r_3}{r_1^2}\right\} \\ &\quad +b_1 - \frac{r_0}{r_1} - y - \frac{r_3 b_2}{r_1} - \frac{r_3}{r_1^2} \end{aligned} \tag{28}$$

It is clear that $C(1,y)$ does not depend on the path from (a_1,a_2) to (b_1,b_2) because the exact form of $\theta_1(t), \theta_2(t)$ as functions of t were never used to derive (28). We state the above derivation as a proposition.

Proposition 2 *Suppose the price function $P(\theta,S)$ and the quantile demand function $q_\tau(\theta,y)$ are defined on connected open sets, and are continuously differentiable on their domain. Suppose all assumptions on preferences stated in assumption 1 are satisfied. Additionally, suppose (19) and (21) hold with $r_1 + r_2 = 0$. Then the compensating variation due to a movement of (θ_1,θ_2) from (a_1,a_2) to (b_1,b_2) for an individual at $\eta = F_\eta^{-1}(\tau)$ is independent of the path along which (θ_1,θ_2) changes, and is given by*

$$\begin{aligned} C(1,y) &= e^{r_1(b_2-a_2)}\left\{\frac{r_0}{r_1} + y + \frac{a_2 r_3}{r_1} + \frac{r_3}{r_1^2} - a_1\right\} \\ &\quad +b_1 - \frac{r_0}{r_1} - y - \frac{r_3 b_2}{r_1} - \frac{r_3}{r_1^2} \end{aligned} \tag{29}$$

An analogous exercise can be done for every other quantiles, which produce different values of the r's in (21) and, correspondingly different values of the compensating variation (29).

Remark 2 *Note that $Q_\tau(y,\theta)$ defined in (10) need not equal the τth quantile of the indirect utility $V(y,\theta,\eta)$ because $V(y,\theta,\eta)$ need not be monotonic in η. Nonetheless, as τ varies over $[0,1]$, we can trace out the distribution of $V(y,\theta,\eta)$. In particular, for each specific quantile, say, $\tau=0.1$ or $\tau=0.5$ etc., we get the value of the compensating variation for individuals with income y and who are at the lowest decile or the median of unobserved heterogeneity, respectively. These need not equal the the lowest decile or median respectively of the marginal distribution of the compensating variation for people with income y.*

The previous proposition can be generalized in the obvious way for general price and quantile functions, as follows.

Proposition 3 *Suppose $\theta\in R^J$; the price function $P(\theta,S)$ and the quantile demand function $q_\tau(\theta,y)$ are defined on connected open sets, and are twice continuously differentiable on their domain. Suppose the Slutsky symmetry condition is satisfied, i.e.*

$$\begin{aligned}&\frac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial\theta_j\partial q}\left\{\frac{\partial q^\tau(y,\theta)}{\partial y}\frac{\partial P(q^\tau(y,\theta),\theta)}{\partial\theta_k}+\frac{\partial q^\tau(y,\theta)}{\partial\theta_k}\right\}\\ =\ &\frac{\partial^2 P(q^\tau(y,\theta),\theta)}{\partial\theta_k\partial q}\left\{\frac{\partial q^\tau(y,\theta)}{\partial y}\frac{\partial P(q^\tau(y,\theta),\theta)}{\partial\theta_j}+\frac{\partial q^\tau(y,\theta)}{\partial\theta_j}\right\}\end{aligned}$$

for all $j\neq k$. Then for any path $\theta(t)$, $t\in[0,1]$, with $\theta(0)=a$ and $\theta(1)=b$, the compensating variation is given by $C(1,y)$, where $C(t,y)$ is the solution to the ordinary differential equation

$$\frac{dC(t,y)}{dt}=\sum_{j=1}^{J}\frac{d\theta_j(t)}{dt}\times\frac{\partial P(\theta(t),q_\tau(\theta(t),y+C(t,y)))}{\partial\theta_j},$$

with initial condition $C(0,y)=0$, and this solution is independent of the path $\theta(t)$.

The proof of this result is completely analogous to that of the previous proposition, and uses the fundamental theorem for line integration that line integrals of gradient fields are path-independent (cf. Courant and John 1989, Sec 1.10, Taylor, 1955 Theorem 13.4.V).

3.3 Multiple Attributes

To incorporate additional hedonic attributes into the above analysis, consider the additively separable utility function

$$U(s,x,c,\eta) = U_1(s,\eta) + U_2(x) + U_3(c)\,, \text{ with } c = y - P(s,x)\,,$$

where s is the key attribute of interest, x represents the other attributes, y is income, and the hedonic price function is given by

$$P(s,x) = P_1(s,\theta) + P_2(x,\delta)$$

We want to measure the distribution of the compensating variation C that solves

$$V(y,\theta,\delta,\eta) = V(y+C,\beta,\delta,\eta)\,,$$

where

$$V(y,\theta,\delta,\eta) = \max_{s,x} U(s,x,y-P_1(s,\theta)-P_2(x,\delta),\eta)$$

If the attributes are continuous and the utility function is differentiable in each, then the first order conditions for maximization are given by

$$\frac{\partial U_1(s,\eta)}{\partial s} - U_3'(y-P_1(s,\theta)-P_2(x,\delta)) \times \frac{\partial P_1(s,\theta)}{\partial s} = 0 \tag{30}$$

$$\nabla_x U_2(x) - U_3'(y-P_1(s,\theta)-P_2(x,\delta)) \times \nabla_x P_2(x,\delta) = 0 \tag{31}$$

while a sufficient second order condition for an interior maximum is that the matrix

$$H = \begin{bmatrix} \begin{array}{c} \frac{\partial^2 U_1(s,\eta)}{\partial s^2} - U_3'' \times \left(\frac{\partial P_1(s,\theta)}{\partial s}\right)^2 \\ +U_3' \times \left(\frac{\partial P_1(s,\theta)}{\partial s}\right)^2 \end{array} & \nabla_x P_2(x,\delta) \times U_3'' \times \frac{\partial P_1(s,\theta)}{\partial s} \\ \nabla_x P_2(x,\delta) \times U_3'' \times \frac{\partial P_1(s,\theta)}{\partial s} & \begin{array}{c} \nabla_{xx} U_2 - U_3' \times \nabla_{xx} P_2(x,\delta) \\ -U_3'' \times \nabla_x P_2(x,\delta) \times \nabla_x P_2(x,\delta)' \end{array} \end{bmatrix} \tag{32}$$

is negative definite for all s, x.

Now, evaluating the first-order conditions (30)-(31) at the optimal choice and differentiating w.r.t. η, we have that

$$\begin{aligned}
\frac{\partial U_1(S^*,\eta)}{\partial s} - U_3'\left(y - P_1(s^*,\theta) - P_2(x,\delta)\right) \times \frac{\partial P_1(S^*,\theta)}{\partial s} &= 0 \\
\left[\begin{array}{l}
\frac{\partial^2 U_1(S^*,\eta)}{\partial s \partial \eta} + \frac{\partial^2 U_1(S^*,\eta)}{\partial s^2}\frac{\partial s^*}{\partial \eta} \\
+U_3''\left(y - P_1(S^*,\theta) - P_2(x,\delta)\right) \times \left\{\frac{\partial P_1(S^*,\theta)}{\partial s}\right\}^2 \times \frac{\partial S^*}{\partial \eta} \\
+U_3''\left(y - P_1(S^*,\theta) - P_2(x,\delta)\right) \times \frac{\partial P_1(x^*,\theta)}{\partial s} \times \nabla_x P_2(x,\delta)\frac{\partial x^*}{\partial \eta} \\
-U_3'\left(y - P_1(S^*,\theta) - P_2(x,\delta)\right) \times \frac{\partial^2 P_1(S^*,\theta)}{\partial s^2}\frac{\partial S^*}{\partial \eta}
\end{array}\right] &= 0
\end{aligned} \tag{33}$$

Similarly

$$\nabla_{xx} U_2(x) \times \frac{\partial x^*}{\partial \eta} - U_3''\left\{\nabla_x P_2(x,\delta)\nabla_x P_2(x,\delta)'\right\}\frac{\partial x^*}{\partial \eta} - U_3''\left\{\nabla_x P_2(x,\delta)\right\}\frac{\partial P_1(s^*,\theta)}{\partial s}\frac{\partial s^*}{\partial \eta} = 0 \tag{34}$$

Equations (33)-(34) can be written in matrix notation as

$$H \times \begin{bmatrix} \frac{\partial s^*}{\partial \eta} \\ \frac{\partial x^*}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 U_1(s^*,\eta)}{\partial s \partial \eta} \\ 0 \end{bmatrix}$$

where H is defined in (32). Therefore,

$$\begin{bmatrix} \frac{\partial s^*}{\partial \eta} \\ \frac{\partial x^*}{\partial \eta} \end{bmatrix} = H^{-1}\begin{bmatrix} -\frac{\partial^2 U_1(s^*,\eta)}{\partial s \partial \eta} \\ 0 \end{bmatrix}$$

implying

$$\frac{\partial S^*}{\partial \eta} = -H^{11} \times \frac{\partial^2 U_1(s^*,\eta)}{\partial s \partial \eta}$$

where H^{11} is the $(1,1)$th entry of the matrix H^{-1}. Now since H is negative definite, so is its inverse. Therefore H^{11} must be strictly negative. Therefore, if $\frac{\partial^2 U_1(S^*,\eta)}{\partial s \partial \eta} < 0$, then it follows that $\frac{\partial V^*}{\partial \eta} > 0$. That is, for given y, θ, δ, we have that $S^*(y,\theta,\delta,\eta)$ is strictly increasing in η. Therefore, we have that for each $\tau \in [0,1]$,

$$S^*\left(y,\theta,\delta,F_\eta^{-1}(\tau)\right) = F^{-1}_{s^*(y,\theta,\delta,\eta)}(\tau),$$

i.e. the value of $S^*(y,\theta,\delta,\eta)$ at the τth quantile of η equals the τth quantile of s^* for fixed values of y,θ,δ.

For measuring the welfare effect of a change in θ, holding δ fixed, we follow the essentially the same steps as outlined above. In particular, we have that

$$V(y,\theta,\delta,\eta) = U_1(s^*,\eta) + U_2(x^*) + U_3(y - P_1(s^*,\theta) - P_2(x^*,\delta))$$

so that, by the envelope theorem, one gets

$$-\frac{\frac{\partial V(y,\theta,\delta,\eta)}{\partial \theta}}{\frac{\partial V(y,\theta,\delta,\eta)}{\partial y}} = \left.\frac{U_3' \times \frac{\partial P_1(s,\theta)}{\partial \theta}}{U_3'}\right|_{s=S^*(y,\theta,\delta),x=x^*(y,\theta,\delta)} = \left.\frac{\partial P_1(s,\theta)}{\partial \theta}\right|_{s=V^*(y,\theta,\delta)}.$$

This last simplification, i.e. that the RHS depends only on $S^*(y,\theta,\delta)$ and not on $x^*(y,\theta,\delta)$, results from the additive separability of the hedonic price function.

Evaluating this at $\eta = F_\eta^{-1}(\tau)$, we get (17) replaced by

$$\frac{\partial Q_\tau(y,\theta,\delta)}{\partial \theta} + \left.\frac{\partial P_1(s,\theta)}{\partial \theta}\right|_{s=F^{-1}_{s^*(y,\theta,\delta)}(\tau)} \times \frac{\partial Q_\tau(y,\theta,\delta)}{\partial y} = 0, \tag{35}$$

where $F^{-1}_{S^*(y,\theta,\delta)}(\tau)$ is the τth quantile of the optimal (i.e. chosen) s across individuals with income y in markets characterized by (θ,δ). Therefore, we can apply the method outlined in the previous subsection, holding δ fixed, and obtain the value of the compensating variation for each type defined by a quantile of η.

4 Empirical Illustration

4.1 Data

The dataset used for our illustration comes from the restricted-access version of Wave 2015 of English Housing Survey (DCLG, 2018), which is a nationally representative survey on housing stock, conditions, and household characteristics. We use the data on rented properties. For each property, we observe the annual rent as well as a range of property characteristics, including floor area, number of floors, dwelling type (terrace, detached, flat, etc.), age, number of bathrooms, bedrooms, and living rooms, and an index of local economic deprivation, measured at the level of the so-called 'Lower Layer Super Output Area'. We also observe a set of property and household characteristics, including structural features of the property (e.g. number of bedrooms, floor area etc.), net annual income of the household, whether the household receives housing benefits, and tenure type, i.e. whether renting privately, via local authority provision, or through housing associations.

To proxy for school quality, we use the average point score per pupil for secondary schools. The point score comes from the Key Stage 4 data in the School Performance Tables, commonly known as *league tables*.[6] These data are publicly available via the UK government's official website, GOV.UK. We exclude independent, i.e. private, schools and schools for children with special educational needs, i.e. special schools, because they follow different admission procedures and cater to a distinct population.

Each property is matched to the nearest school based on postcode proximity. The matching process was carried out using the open-source geographic information system software, QGIS. We start with a total of 6,611 property-school matched observations. We then exclude 110 cases where household incomes are negative after accounting for rent. To remove the outliers, we further drop the households with rent-to-income ratios above the 95th percentile and those within the top or bottom 5 percent of the income distribution, leading to excluding 866 observations. The final sample includes 5,635 properties, each matched to the nearest school.

Table A.1 of the Appendix presents the descriptive statistics for the dataset. On average, households in our sample have a post-tax weekly income of £396, with around £111 allocated to rent. Around 28 percent of households rent privately, while the remainder are social renters, either from local authorities or housing associations. Approximately 56 percent of the respondents receive housing benefits, and 55 percent reside in areas within the three most deprived deciles. For the purpose of empirical application, we reduce the dimensionality of the property and household characteristics by using its first principal component.[7]

[6] Key Stage 4 represents the two years of education for students aged 14 to 16, corresponding to Years 10 and 11 in the English education system.

[7] Appendix Table A.2 reports the correlations between the original variables, while Appendix Table A.3 provides the loadings on the first principal component.

Table 1 presents the results of the hedonic regression for property rental prices for the whole of England. The rents are positively and significantly associated with the quality of the nearest school: an increase in one standard deviation in school quality is associated with extra £7.5 or 7 percent of weekly rent.[8] The literature on the relationships between school quality and rental prices is scarce, making it hard to compare our result to earlier findings.[9] Our estimate is at the upper bound of what is typically found in the larger literature that focuses on purchase property prices (Machin, 2011).

4.1.1 Computation Steps

The computation of welfare is done through the following steps, where for simplicity, we use $\tau = 0.5$ for illustration.

1. Construct the scalar index X which equals the first principal component of all non-S attributes (STATA command pca). This is done to reduce the dimension of the covariates.

2. Divide locations into M markets. For each market, estimate the price function by regressing price of unit on $\ln S$ and X; call the coefficients $\alpha_{1m}, \alpha_{2m}, \delta_m$, $m = 1, ...M$

$$P_{mi}(S, X) = \alpha_{1m(i)} + \alpha_{2m(i)} \ln S_i + \delta_{m(i)} X_{m(i)}$$

3. Run a linear median regression (qreg in STATA), using all observations, of $\ln S_i$ on an intercept and $y_i, \alpha_{1m(i)}, \alpha_{2m(i)}$ and $\delta_{m(i)}$

$$med\left(\ln S_i | y_i, \alpha_{1m(i)}, \alpha_{2m(i)}, \delta_{m(i)}\right) = r_0 + r_1\left(y_i - \alpha_{1m(i)}\right) + r_3 \alpha_{2m(i)} + r_4 \delta_{m(i)}$$

where $m(i)$ is the market in which i lives

[8] One standard deviation of logarithm of total average point score per pupil equals 0.20.

[9] To the best of our knowledge, Bayer, Ferreira, and McMillan (2007) is the only paper that includes an analysis of rental prices. They use a dataset that combines rental and purchase prices to study the association with elementary school quality. They found that households are willing to pay less than 1 percent more in house prices when the average school performance increases by 5 percent. We estimate a moderately higher relationships of 1.7 percent, focusing solely on secondary schools and renters.

4. Fix a value of $y = y_0$, $\delta = \delta_0$ (say, median values of y and δ in the data)

5. Then consider the change in α_1, α_2 from (a_1, a_2) to (b_1, b_2) (say, from the bottom quartile to top quartile)

6. Calculate compensating variation as

$$\begin{aligned} CV &= e^{r_1(b_2-a_2)}\left\{\frac{r_0}{r_1} + y_0 + \frac{a_2 r_3}{r_1} + \frac{r_3}{r_1^2} - a_1\right\} \\ &\quad + b_1 - \frac{r_0}{r_1} - y_0 - \frac{r_3 b_2}{r_1} - \frac{r_3}{r_1^2} \end{aligned}$$

7. Replace median in Step 3 to other quantiles, e.g. $\tau = 0.25, 0.75$ etc. and repeat steps 4-6.

4.2 Results

In this section, we report the results obtained by applying the methods outlined in Section 4.1.1 with a single attribute, viz. school-quality. For the estimation we split the dataset into 9 markets, represented by English regions: North, East Yorkshire and the Humber, North West, East Midlands, West Midlands, South West, East England, and South East London. Table 2 presents the results of hedonic regressions for rental prices estimated for different regions. The model is described in Step 2 of Section 4.1.1. The estimates for the relationship between rental prices and school quality vary from 2.92 in North East to 40.48 in London.

Table 3 reports the results of a linear quantile regression for the logarithm of the school quality presented in Step 3 of Section 4.1.1. To control for endogeneity, the term $y - \alpha_{1m(i)}$ in the model is instrumented with $s - \alpha_{1m(i)}$, where s is the level of past year savings reported by the household, and using the 'ivqreg' command in STATA, implementing the method of Chernozhukov and Hansen 2004. The model produces the following results. The estimate of r_1 is positive and decreasing across quartiles: income affects the demand for school quality less for the households who value schools more. The estimate of r_3 is negative and increasing from 25th to 75th percentile, *i.e.* its magnitude decreases: when the relationships between school quality and prices grow stronger, demand decreases less for those who value schools more. Both coefficient are significant for the 25th and 50th percentile, but not for the 75th percentile of the distribution of school quality.

We now introduce a hypothetical policy change that increases the sensitivity of rental prices to school quality. An example of such policy would be introducing more strict distance criteria for school admission. In particular, we change the relationships between the school quality and rental prices, $\alpha_{2m(i)}$, from the 25th percentile of the distribution across the markets (18.30 in the North West) to the 75th percentile (28.56 in the East), and analogously exchange the constant term. By design, rental prices near better schools should increase, while the prices near worse schools should go down. Figure 2 presents the change in the relationship between school quality and rental payments as a result of the policy change. In our sample, rents increase for the majority of school quality levels, with only houses near the worst schools becoming cheaper to rent.

The resulting estimates of compensating variation for different quantiles of η evaluated at specific quantiles of income are presented in Table 4 and Figure 3. The policy change results in a universal welfare loss. Compensating variation at the median preference for school quality ($\eta = F_{\eta}^{-1}(0.5)$), evaluated at y_0 equal to the median income, equals £14, which constitutes 13 percent of average weekly rent in England. Households at the same income that live near schools of higher quality (i.e. have higher η) experience a greater welfare loss that those living near worse schools. This is to be expected, as their rents increase more, see Figure 2. For households at identical percentiles of η, those with higher incomes experience higher loss. This is again to be expected, as comparatively richer households are likely to live near better schools, where housing costs rise after the policy change, leaving lesser funds for consumption.

As a robustness check, we also estimated the compensating variation for deciles of preference for school quality between the 20th and 80th percentiles, and find that the same patterns hold for the compensating variation. These additional results for deciles are presented in Table 5 and Figure 4.

5 Conclusion

In this paper, we analyze individual demand of attributes in markets characterized by smooth, nonlinear budget constraints. We provide an econometric method of computing welfare effects of policy interventions that change the budget frontier. The method works by deriving the analog of Roy's identity when preferences are nonsatiated and budget-constraints are nonlinear. This analog takes the form of a system of PDEs that involve partial derivatives of the indirect utility function with respect to income and with respect to parameters characterizing the price function. We show how dimension restrictions on unobserved heterogeneity and a single-crossing property of preferences enable one to identify the coefficient functions in these PDEs, and then derive and use a set of Slutsky-like symmetry conditions to calculate welfare effects resulting from a change in the budget frontier. We provide a practical illustration of our methods to evaluate welfare effects of a hypothetical change in relationship between property rent and neighboring school quality in England.

Two issues are left to future work. The first is to develop methods for flexible calculation of the budget frontier, correcting for potential omitted variable bias, and possibly using penalized regression with many covariates. The second is to develop methods of inference for the estimated welfare thereof.

Tables

Table 1: Hedonic regression results for England.

	Weekly rent
Logarithm of total average point score per pupil	37.384***
	(2.752)
First principal component of property characteristics	3.977***
	(0.279)
Constant	-125.187***
	(16.243)
R-squared	0.06
Number of observations	5,635

Note: Standard errors in parentheses. * p<0.10, ** p<0.05, *** p<0.01

Table 2: Hedonic regression results by region.

	North East	North West	Yorkshire and the Humber	E. Midlands	W. Midlands	East	London	South East	South West
Logarithm of school quality	2.927	18.297***	16.274***	25.836***	18.563***	28.556***	40.484**	26.747***	30.954***
	(3.173)	(4.857)	(4.626)	(6.574)	(6.021)	(7.064)	(16.111)	(7.858)	(9.162)
First principal component	4.451***	4.035***	5.239***	5.249***	4.880***	5.900***	8.911***	5.238***	5.216***
	(0.527)	(0.436)	(0.512)	(0.616)	(0.522)	(0.684)	(1.262)	(0.801)	(0.761)
Constant	53.490***	-28.164	-25.139	-76.485**	-30.314	-73.695*	-119.299	-52.004	-92.781*
	(18.361)	(28.671)	(27.252)	(38.342)	(35.359)	(41.310)	(96.190)	(46.566)	(54.412)
R2	0.16	0.10	0.15	0.15	0.14	0.13	0.07	0.07	0.10
Observations	385	829	666	497	596	681	672	782	527

Note: Standard errors in parentheses. * p<0.10, ** p<0.05, *** p<0.01

Table 3: Quantile instrumental variable regression estimates for school quality demand at different quartiles.

	25th percentile	50th percentile	75th percentile
$y - \alpha_{1m(i)}$	0.00066**	0.00047***	0.00022
	(0.00027)	(0.00016)	(0.00017)
$\alpha_{2m(i)}$	-0.00401*	-0.00252**	-0.00105
	(0.00211)	(0.00122)	(0.00135)
$\delta_{m(i)}$	0.01553***	0.01136***	0.01119***
	(0.00464)	(0.00337)	(0.00293)
Constant	5.46753***	5.66965***	5.85488***
	(0.07841)	(0.04600)	(0.04970)
Observations	5635	5635	5635

Note: To control for endogeneity, the term $y - \alpha_{1m(i)}$ in the model is instrumented with $s - \alpha_{1m(i)}$, where s is the level of savings reported by the household. F-statistics for the first stage from a linear regression model for the mean equals 47.97. Standard errors in parentheses. * p<0.10, ** p<0.05, *** p<0.01

Table 4: Compensating variation estimates for quartiles of preference for school quality, GBP.

	Income percentile		
School quality percentile	25th percentile	50th percentile	75th percentile
25th percentile	11.8520	12.4455	13.2846
50th percentile	13.6383	14.0606	14.6575
75th percentile	15.0347	15.2322	15.5114

Table 5: Compensating variation estimates for deciles of preference for school quality, GBP.

	Income percentile		
School quality percentile	25th percentile	50th percentile	75th percentile
20th percentile	11.6858	12.2037	12.9359
30th percentile	12.2099	12.7434	13.4976
40th percentile	13.0697	13.4612	14.0146
50th percentile	13.6383	14.0606	14.6575
60th percentile	14.1730	14.5000	14.9621
70th percentile	14.7370	14.9303	15.2035
80th percentile	15.4711	15.6454	15.8919

Figures

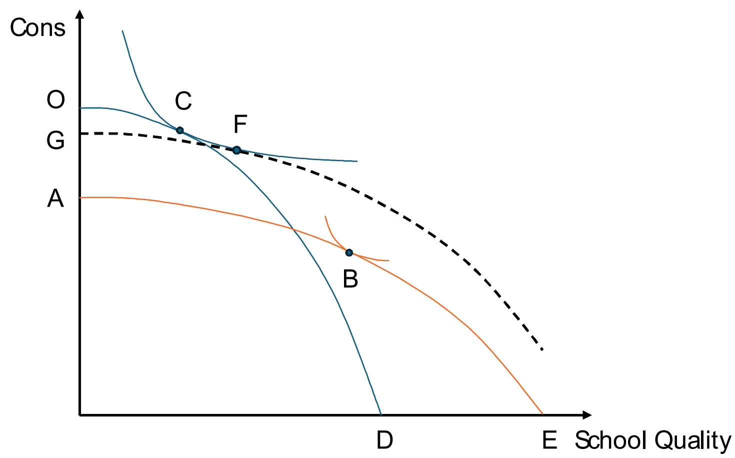

Figure 1: Compensating variation when the budget frontier changes.

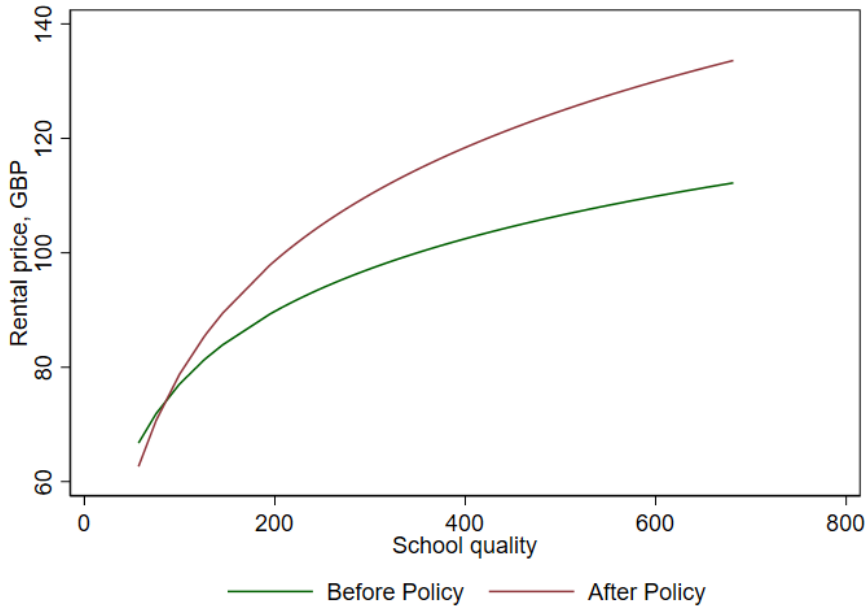

Figure 2: Change in the relationships between rents and school quality as a result of policy change.

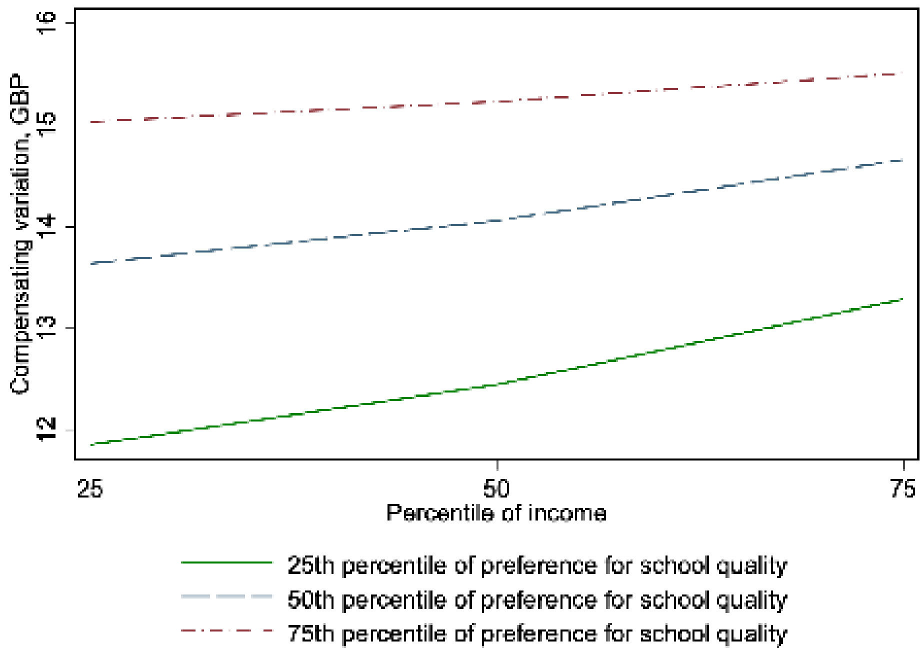

Figure 3: Compensating variation estimates for quartiles of preference for school quality, GBP.

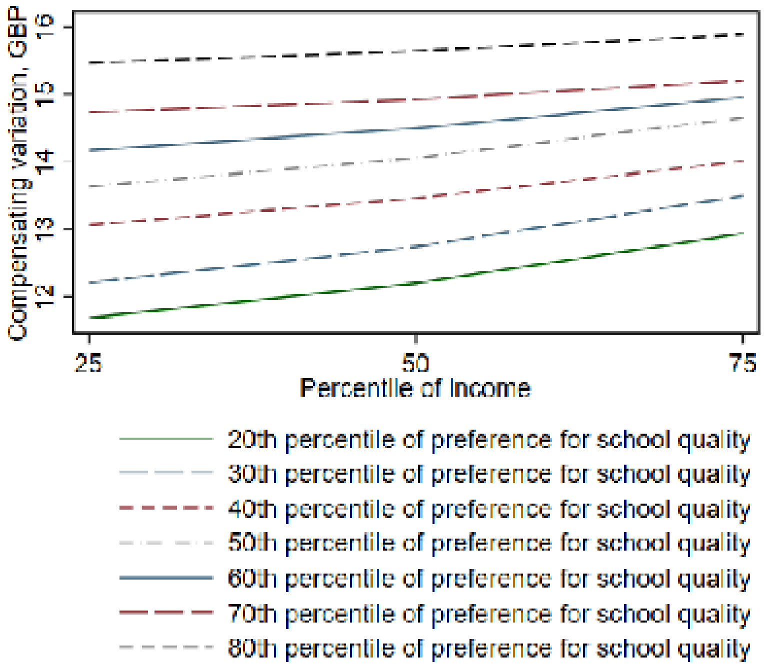

Figure 4: Compensating variation estimates for deciles of preference for school quality, GBP.

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Appendix

A Supplementary Material

Table A.1: Descriptive statistics for matched property-school observations.

Variable	Mean	SD
Total weekly rent	111.17	42.62
Net household income including savings and benefits	396.37	146.72
Total average point score per pupil	362.65	67.30
Logarithm of total average point score per pupil	5.88	0.20
Floor area, sqm	65.85	20.91
Number of floors:		
1	0.10	0.30
2	0.67	0.47
3	0.13	0.34
4	0.04	0.21
5 or more	0.05	0.21
Dwelling type:		
end terrace	0.12	0.33
mid terrace	0.22	0.41
semi detached	0.20	0.40
detached	0.01	0.12
bungalow	0.10	0.30
converted flat	0.04	0.19
purpose built flat, low rise	0.27	0.45
purpose built flat, high rise	0.03	0.16
Dwelling age:		
pre 1850	0.01	0.10
1850 to 1899	0.05	0.22
1900 to 1918	0.06	0.23
1919 to 1944	0.13	0.34
1945 to 1964	0.27	0.45
1965 to 1974	0.17	0.37
1975 to 1980	0.07	0.26
1981 to 1990	0.08	0.28
1991 to 2002	0.08	0.27
post 2002	0.07	0.25
Number of bedrooms:		
1	0.23	0.42
2	0.37	0.48
3	0.36	0.48

Table A.1: Descriptive statistics for matched property-school observations.

Variable	Mean	SD
4	0.04	0.19
5 or more	0.01	0.07
Number of living rooms:		
1	0.01	0.08
2	0.88	0.32
3	0.10	0.30
4	0.00	0.07
5 or more	0.00	0.04
Number of bathrooms:		
1	0.95	0.21
2	0.04	0.20
3 or more	0.00	0.06
Tenure type:		
Private rented	0.28	0.45
Local Authority	0.30	0.46
Housing Association	0.42	0.49
Housing benefits:		
Yes	0.56	0.50
No	0.44	0.50
Deprivation decile:		
1 - Most deprived	0.23	0.42
2	0.18	0.38
3	0.14	0.35
4	0.12	0.32
5	0.09	0.28
6	0.08	0.27
7	0.06	0.24
8	0.05	0.22
9	0.04	0.19
10 - Least deprived	0.02	0.14
Region:		
North East	0.07	0.25
North West	0.15	0.35
Yorkshire and the Humber	0.12	0.32
East Midlands	0.09	0.28
West Midlands	0.11	0.31
East	0.12	0.33
London	0.12	0.32
South East	0.14	0.35
South West	0.09	0.29
Observations	5635	

Table A.2: Correlation between floor area and other property characteristics.

Variable	Floor area, sqm
Floor area squared	0.929
Number of floors:	
2	0.196
3	0.002
4	-0.036
5 or more	-0.077
Dwelling type:	
mid terrace	0.213
semi detached	0.229
detached	0.243
bungalow	-0.221
converted flat	-0.117
purpose built flat, low rise	-0.361
purpose built flat, high rise	-0.050
Dwelling age:	
1850 to 1899	0.044
1900 to 1918	0.058
1919 to 1944	0.052
1945 to 1964	0.031
1965 to 1974	-0.044
1975 to 1980	-0.066
1981 to 1990	-0.134
1991 to 2002	-0.022
post 2002	0.064
Number of bedrooms:	
2	-0.124
3	0.434
4	0.353
5 or more	0.228
Number of living rooms:	
2	-0.273
3	0.282
4	0.147
5 or more	0.014
Number of bathrooms:	
2	0.275
3 or more	0.148
Deprivation decile:	
2	-0.021
3	-0.010

Table A.2: Correlation between floor area and other property characteristics.

Variable	Floor area, sqm
4	-0.010
5	0.008
6	0.021
7	-0.015
8	-0.025
9	0.024
10 - Least deprived	0.008
Tenure type:	
Local Authority	-0.051
Housing Association	-0.048
Housing benefits:	
No	0.091
Observations	5635

Table A.3: Principal component loadings and unexplained variance for property characteristics.

Variable	Loadings on PC1	Unexplained Variance
Floor area, sqm	0.4024	.371
Floor area - squared, sqm	0.3672	.4762
Number of floors:		
2	0.2669	.7233
3	-0.1029	.9589
4	-0.0940	.9657
5 or more	-0.1235	.9407
Dwelling type:		
mid terrace	0.1535	.9085
semi detached	0.2119	.8257
detached	0.1521	.9102
bungalow	-0.1457	.9175
converted flat	-0.0477	.9911
purpose built flat, low rise	-0.2737	.709
purpose built flat, high rise	-0.0952	.9648
Dwelling age:		
1850 to 1899	0.0430	.9928
1900 to 1918	0.0626	.9848
1919 to 1944	0.1004	.9609
1945 to 1964	0.0264	.9973
1965 to 1974	-0.0813	.9743
1975 to 1980	-0.0612	.9854
1981 to 1990	-0.0739	.9788
1991 to 2002	-0.0088	.9997
post 2002	-0.0179	.9988
Number of bedrooms:		
2	-0.1708	.8867
3	0.3083	.6308
4	0.1599	.9007
5 or more	0.0944	.9654
Number of living rooms:		
2	-0.2834	.6881
3	0.2790	.6977
4	0.1043	.9577
5 or more	0.0168	.9989
Number of bathrooms:		
2	0.1453	.918
3 or more	0.0660	.9831
Deprivation decile:		
2	-0.0379	.9944

Table A.3: Principal component loadings and unexplained variance for property characteristics.

Variable	**Loadings on PC1**	**Unexplained Variance**
3	-0.0101	.9996
4	0.0064	.9998
5	0.0069	.9998
6	0.0198	.9985
7	0.0080	.9998
8	-0.0021	1
9	0.0169	.9989
10 - Least deprived	0.0163	.999
Tenure type:		
Local Authority	-0.0529	.9891
Housing Association	-0.0384	.9943
Housing benefits:		
No	0.0707	.9806
Observations	5635	

Table A.4: Regression of Rental Price on School Quality and Housing Characteristics

Variable name	Est.	SE
Logarithm of total average point score per pupil	5.193**	(2.073)
Floor area, sqm	-0.148**	(0.071)
Floor area squared	0.001***	(0.000)
Number of floors		
1 (reference)	.	(.)
2	-17.089	(11.948)
3	-16.826	(12.013)
4	-13.656	(12.131)
5 or more	-20.251*	(12.301)
Dwelling type		
end terrace (reference)	.	(.)
mid terrace	-0.117	(1.409)
semi detached	0.337	(1.445)
detached	0.111	(3.647)
bungalow	-17.355	(11.990)
converted flat	2.430	(2.648)
purpose built flat, low rise	1.279	(1.713)
purpose built flat, high rise	13.560***	(3.992)
Dwelling age		
pre 1850 (reference)	.	(.)
1850 to 1899	-1.883	(4.118)
1900 to 1918	3.011	(4.116)
1919 to 1944	8.062**	(4.006)
1945 to 1964	6.368	(3.959)
1965 to 1974	6.773*	(4.000)
1975 to 1980	8.251**	(4.143)
1981 to 1990	9.330**	(4.108)
1991 to 2002	11.459***	(4.111)
post 2002	14.263***	(4.152)
Number of bedrooms		
1 (reference)	.	(.)
2	10.832***	(1.316)
3	19.304***	(1.786)
4	30.049***	(2.906)
5 or more	20.708***	(6.005)
Number of living rooms		
1 (reference)	.	(.)
2	7.406	(4.800)
3	10.077**	(5.000)
4	7.427	(7.735)

Table A.4: Regression of Rental Price on School Quality and Housing Characteristics

Variable name	Est.	SE
5 or more	18.043*	(10.044)
Number of bathrooms		
1 (reference)	.	(.)
2	12.239***	(2.111)
3 or more	26.237***	(6.907)
Tenure type		
Private rented (reference)	.	(.)
Local Authority	-50.678***	(1.205)
Housing Association	-40.394***	(1.078)
Housing benefits		
Yes (reference)	.	(.)
No	-3.527***	(0.819)
Deprivation decile		
1 - Most deprived (reference)	.	(.)
2	-0.095	(1.255)
3	0.911	(1.368)
4	4.375***	(1.467)
5	8.783***	(1.619)
6	9.549***	(1.692)
7	11.709***	(1.842)
8	9.402***	(2.041)
9	14.171***	(2.315)
10 - Least deprived	14.833***	(2.927)
Region		
North East (reference)	.	(.)
North West	1.923	(1.816)
Yorkshire and the Humber	-1.820	(1.880)
East Midlands	1.418	(2.002)
West Midlands	7.208***	(1.922)
East	19.636***	(1.905)
London	59.699***	(2.021)
South East	28.304***	(1.888)
South West	9.916***	(2.012)
Constant	88.135***	(18.272)
R2	0.54	
Observations	5635	

Note: Standard errors in parentheses. * $p<0.10$, ** $p<0.05$, *** $p<0.01$